Title: Simulation of Nonstationary Poisson Processes

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Short Description:
Nonstationary Poisson processes are often used to model time-dependent arrival rates $\lambda(t)$ of customers for real-world systems. The inverse transform method is an efficient method to simulate customer arrival times. While this is very easy for Poisson processes with constant rate $\lambda(t) = \lambda$, we point to some problems for nonstationary Poisson processes.

Keywords: Nonstationary Poisson Process, generate arrivals, inverse transform

Example:
The stationary Poisson process has a constant rate $\lambda > 0$ of customer arrivals. This means the interarrival time $A$ between two consecutive customers is an exponential random variable with the cumulative distribution function $A(t) = 1 - e^{-\lambda t}$. In order to generate the times of arrivals $t_1, t_2, \ldots$ of customer $1, 2, \ldots$ starting from time $t_0$, we use the inverse transform $A^{-1}$ recursively.

Algorithm 1: Generating a Poisson process
1. Generate random variate $t$ from uniform distribution on interval $[0; 1]$: $t \sim \text{Uniform}(0, 1)$.
2. Return the $i$-th customer arrival: $t_i = t_{i-1} + A^{-1}(t) = t_{i-1} - \frac{1}{\lambda} \log t$.

Generating arrival times that follow a nonstationary Poisson process with $\lambda(t)$ appears to be as easy. However, substituting $\lambda(t_{i-1})$ in step 2 of algorithm 1 for $\lambda$ is incorrect. An illustration is given in Figure 1. The red line shows the given arrival rate $\lambda(t)$, while the blue bars indicate the simulated arrival rate $\tilde{\lambda}(t)$. The substitution skips large intervals at $t_{i-1}$ independent of the following rates, if the arrival rate $\lambda(t_{i-1})$ is low. E.g. for $t_{i-1} = 14$, the arrival rate is $\lambda(t_{i-1}) = 0.02$ and the mean interarrival time is $1/\lambda(t_{i-1}) = 50$. Hence, the next arrival $t_i$ will be generated at $t_{i-1} + 1/\lambda(t_{i-1}) = 64$ on average and the upcoming peaks in the arrival rate are missed. Accordingly, the high arrival density between $t = 27$ and $t = 67$ is not simulated.

![Figure 1: Substituting $\lambda(t_{i-1})$ in step 2 of Algorithm 1 for $\lambda$ is incorrect to generate a nonstationary Poisson process](image-url)
Therefore, the expectation function $\Lambda$ is used to generate customer arrivals for the nonstationary Poisson process. $\Lambda(t)$ is the expected number of arrivals in the interval $[0; t]$. It is

$$\Lambda(t) = \int_0^t \lambda(x)dx.$$ 

We first generate Poisson arrival times $y_1, y_2, \ldots$ with rate 1 and get the customer arrivals of the nonstationary Poisson process by $t_i = \Lambda^{-1}(y_i)$. The inverse $\Lambda^{-1}$ of the expectation function has to be derived therefore. Then, we have the following algorithm.

**Algorithm 2**: Generating a nonstationary Poisson process (inverse transform method)

1. Generate random variate $t$ from uniform distribution on interval $[0; 1]$: $t \sim \text{Uniform}(0, 1)$.
2. Generate arrival times with rate 1: $y_i = y_{i-1} - \log(t)$.
3. Return $i$-th customer arrival of nonstationary Poisson process: $t_i = \Lambda^{-1}(y_i)$.

Figure 2(a) shows how this algorithm works. We use the same arrival rate $\lambda(t)$ as in the example before. The blue curve depicts the expectation function of the number of arrivals. On the y-axis, arrivals $y_i$ of a Poisson process with rate 1 are generated. Hence, the number of arrivals corresponds to the expectation function $\Lambda$. The inverse $\Lambda^{-1}$ returns then the arrivals $t_i$ of the nonstationary Poisson process on the x-axis. The generation of $t_i$ does not depend from the rate $\lambda(t_i-1)$ and the rise of the arrival rate at time $t = 27$ is not missed. The simulated arrival rates when applying algorithm 2 is shown in Figure 2(b). In that case, the simulated arrival rates agree with $\lambda(t)$. This method is very efficient, as for each generated $y_i$, the corresponding customer arrival $t_i$ is determined. This is an advantage to other approaches like the acceptance-rejection method.

However, the inversion of $\Lambda$ is required which might be difficult. In addition, the value range of $\Lambda$ has to be taken into account in algorithm 2. We consider for example the arrival rate $\lambda(t) = e^{-\alpha t}$. The expectation function is $\Lambda(t) = -\frac{1}{\alpha} (e^{-\alpha t} - 1)$ and its inverse $\Lambda^{-1}(y) = -\frac{\log(-\alpha y + 1)}{\alpha}$. If we now generate the arrival rates $y_i$ according to a Poisson process with rate 1, $y_i$ is not bounded in contrast to the expectation function with $\lim_{t \to \infty} \Lambda(t) = \frac{1}{\alpha}$. Considering the algorithm 2 for generating random variates of the nonstationary Poisson process, this means that no more arrival ($t_i = \infty$) has to be generated, if $y_i$ exceeds $\frac{1}{\alpha}$. Another possibility to generate a nonstationary Poisson process is the forward integral method as described in [2]. Figure 3(a) illustrates the derivation of this method. It shows the time dependent arrival rate $\lambda(t)$. We consider an arrival event at time $t_0$ and want to generate the next customer arrival. The time interval from $t_0$ until $t_1$ is discretized into $n$ intervals of
length $\Delta t$. If we assume the arrival rate $\lambda(t)$ to be constant during $\Delta t$, then the $i$-th interval has the arrival rate

$$
\lambda_i = \lambda [t_0 + (i - 1)\Delta t] \quad \text{for} \quad i = 1, 2, \cdots, n .
$$

The probability that there is no arrival during the $i$-th interval is $P_i = e^{-\lambda_i \Delta t}$. Thus, the probability that no arrival occurs during the interval $(t_0, t)$ is the product of all individual probabilities which are statistically independent. The random variable $T_{A,t_0}$ denotes the interarrival time of the known arrival event at $t_0$ and the next arrival:

$$
P(T_{A,t_0}) = \prod_{i=1}^{n} P_i = \prod_{i=1}^{n} e^{-\lambda_i \Delta t} = e^{-\Delta t \prod_{i=1}^{n} \lambda_i} .
$$

With $n \rightarrow \infty$ and $\Delta t \rightarrow 0$, the limit of $P(T_{A,t_0})$ is the complementary cumulative distribution function (CCDF) of the interarrival times at time $t_0$:

$$
F_{t_0}^{c}(t_A) = \lim_{n \rightarrow \infty, \Delta t \rightarrow 0} e^{-\Delta t \prod_{i=1}^{n} \lambda_i} = e^{-\int_{t_0}^{t_A} \lambda(t)dt} .
$$

This allows to formulate the algorithm 3 for generating a nonstationary Poisson process. An example is depicted in Figure 3(b). Similar to algorithm 2, for each customer arrival a definite integral and the inverse function at a particular argument has to be computed numerically. Nevertheless, both algorithms can be efficiently implemented due to the generation of a single random value for every customer arrival.

Algorithm 3: Generating a nonstationary Poisson process (forward integral method)

1. Generate random variate $z$ from uniform distribution on interval $[0; 1]$: $z \sim \text{Uniform}(0, 1)$.
2. Return $i$-th customer arrival of nonstationary Poisson process: $t_i = t_{i-1} + F_{t_{i-1}}^{c}^{-1}(z)$.

References:
